Kähler representations for twisted supergravity

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Introduction

The notion of twisting a supersymmetry algebra is very useful to understand (i) its structure and (ii) the properties of the corresponding quantum field theory.

A generic definition of the supersymmetric twist is that it is a mere linear change of field variables that reduces in a controlable way the size of the global symmetries, but keeps it large enough to uniquely determine the same theory as the superPoincaré symmetry.

One can for instance covariantly select among the 16 generators of maximal supersymmetry $n \leq 9$ susy generators that close "off-shell".

Then many of all known finiteness properties can be proven within the context of local quantum field theory. One finds that maximal supersymmetry is overdetermining,

The twisted fields look like those of a Topological Field Theory : for instance, in d=4, the 8 components of a couple of Majorana spinors are expressed as a vector, a selfdual 2-form and a scalar.

The spin-statistics relation is recovered after untwisting, and the full Poincaré supersymmetry appears as an effective phyical symmetry. The missing (eg 7=16-9 for N=4 SYM) symmetry generators pop-up for free, without having been invited.

Analogous features occur for conformal supersymmetry.

Supersymmetry and holomorphic twist

The idea is of considering manifolds with $SU(n) \subset SO(d = 2n)$ holonomy, and of using holomorphic representations of spinors.

This gives a strikingly simple description of N=1 SYM supersymmetric theories, up to d=10.

The trick is that the 2^n components of a spinor naturally decompose on holomorphic and antiholomorphic forms.

The new result is for N=1, d=4 supergravity.

We first explain the d=4 Yang–Mills case.

The action is

$$\int d^4xtr\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\not{D}\lambda + \frac{1}{2}h^2\}$$

$$\int d^4xtr\{\frac{1}{2}\bar{\varphi}_{D_\mu D^\mu}\varphi - i\bar{\lambda}'\not{D}\lambda' + \frac{1}{2}X\bar{X}\}$$

When the 4-manifold has holonomy $SU(2)$, both balanced twisted multiplets (3,4,1) and (2,4,2) of the N=1,d=4 gauge theory are

$$A_m,\ A_{\bar{m}},\ \Psi_m,\ \kappa_{\bar{m}\bar{n}},\ \kappa,\ h$$

$$\Phi,\ \bar{\Phi},\ \Psi_{\bar{m}},\ \kappa_{mn},\ \bar{\kappa},\ B_{\bar{m}\bar{n}},\ B_{mn}$$

The complex coordinate notation is $z^m, z^{\bar{m}}$ with $m = 1, 2$ and $\bar{m} = \bar{1}, \bar{2}$ for the euclidean space.

To build invariants, one uses the complex structure tensor J.

The notation $X_m Y_{\bar{m}}$, m=1,2, means

$$X_m Y_{\bar{m}} \equiv X_m Y_{\bar{n}} J^{m\bar{n}}$$

where the tensor $J^{m\bar{n}}$ is the complex structure of the Kähler manifold with $SU(2) \times U(1) \subset SO(4)$ holonomy.

One has

$$F_{\mu\nu}F^{\mu\nu} + d(..) = F_{\bar{m}\bar{n}}F_{mn} + (F_{m\bar{m}})^2$$
$$\sim F_{mn}F^{mn} + hF_m^m + \frac{1}{2}h^2$$

Moreover, selfduality of a 2-form is

$$F_{\bar{m}\bar{n}} = F_{mn} = 0 \qquad F_m^m = J^{m\bar{n}}F_{m\bar{n}} = 0$$

and antiselfduality is

$$F_{m\bar{n}} + J_{m\bar{n}}F_p^p = 0$$

Twisted spinors

In Euclidean 4-space, the "analytical continuation of a Majorana spinor" is defined by (anti) holomorphic forms

$$\lambda \sim (\Psi_m,\ \kappa_{\bar{m}\bar{n}},\ \kappa)$$

that is, $(\frac{1}{2},\frac{1}{2}) = (1,0) \oplus (0,2) \oplus (0,0)$**. This generalizes in higher dimensions.**

Indeed, [lawson], on a complex spin manifold the complex spinors can be identified with forms $S_\pm \otimes \mathbb{C} \sim \Omega^{0,{even \atop odd}}$**, so that we can identify** Ψ_m **as a left–handed Weyl spinor** λ_α **and** $(\kappa_{\bar{m}\bar{n}}, \kappa)$ **as a right–handed Weyl spinor** $\bar{\lambda}^{\dot{\alpha}}$**. A reality condition implies that the fields** $\Psi_m,\ \kappa_{\bar{m}\bar{n}},\ \kappa$ **count for 4 (real) field degrees of freedom. This is what one enforces in the path integral formula.**

What is going on is the following change of field variables $(\Psi_m, \kappa_{\bar{m}\bar{n}}, \kappa)$, with

$$\Psi_m = \lambda^\alpha \sigma_{\mu\ \alpha 1} e^\mu_m \ ,$$
$$\kappa_{\bar{m}\bar{n}} = \bar{\lambda}_{\dot{\alpha}}\ \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{2}}\ e^\mu_{\bar{m}} e^\nu_{\bar{n}} \ ,$$
$$\kappa = \delta_{\dot{2}}{}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}$$

We have defined the euclidean σ matrices as $\sigma_\mu = (i\tau^c, \mathbf{1})$, $\tau^c,\ c = 1, 2, 3$ being the Pauli matrices.

There is nothing very mysterious. In a Weyl basis (where γ^5 is diagonal), one has something like

$$\Psi_m \sim \lambda^\alpha$$

$$\kappa_{\bar{m}\bar{n}} = \lambda_{\dot{1}} + \lambda_{\dot{2}} \qquad \kappa = \lambda_{\dot{1}} - \lambda_{\dot{2}}$$

Thus the 4d-Dirac action for a Majorana spinor satisfies

$$\bar{\lambda}\gamma^{\mu} D_{\mu}\lambda = \kappa_{\bar{m}\bar{n}} D_{[m}\Psi_{n]} + \kappa D_{\bar{m}}\Psi_{m}$$

It can be added to the Yang-Mills action

$$\int F_{mn}F^{mn} + hF^{m}_{m} + \frac{1}{2}h^{2}$$

These decomposition is most useful for twisting YM theories with only N=1 susy (Johansen, Losev, Anselmi, Witten, Baulieu, Kanno, Singer, Bossard, Tanzini, etc...).

3 nilpotent symmetry generators exist (one scalar δ and one vector $\delta_{\bar{p}}$), acting on the fields of the balanced multiplets, with

$$\begin{aligned} \delta^2 &= 0 \\ \{\delta_{\bar{p}}, \delta_{\bar{q}}\} &= 0 \\ \{\delta, \delta_{\bar{p}}\} &= \partial_{\bar{p}} + \delta^{gauge}(A_{\bar{p}}) \end{aligned}$$

The δ and $\delta_{\bar{p}}$ transformation laws can be determined, as when one solves cohomological BRST equations.

This reduces the question of finding a supersymmetric theory to a cohomological problem.

When one writes an invariant Lagrangian, it turns out to be δ-exact. Then, the vector symmetry $\delta_{\bar{p}}$ is associated to the conservation law of the δ-antecedent of the energy momentum tensor of the δ-exact action.

So, N=1,d=4 supersymmetry is fully fixed by the invariance under 3 generators.

The fourth symmetry is a consequence of these 3 symmetries. Its generator δ_{mn} is also nilpotent.

In practice, the 3 relevant supersymmetry generators δ and $\delta_{\bar{p}}$ can be determined either directly, by TQFT gauge symmetry considerations, as a generalization of Baulieu–Singer method, or by trial and error, by enforcing the nilpotency properties.

The scalar supersymmetry of N=1,d=4 possesses a very simple cohomology. More fields are needed to enrich this cohomology, as in N=2, or N=4.

The action of δ and $\delta_{\bar{p}}$ reads as follows, in a TQFT typical way :

$$\delta A_m = \Psi_m \qquad \delta_{\bar{p}} A_m = J_{\bar{p}m}\kappa$$

$$\delta A_{\bar{m}} = 0 \qquad \delta_{\bar{p}} A_{\bar{m}} = \kappa_{\bar{p}\bar{m}}$$

$$\delta \Psi_m = 0 \qquad \delta_{\bar{p}} \Psi_m = F_{\bar{p}m} - J_{\bar{p}m} h$$

$$\delta \kappa = h \qquad \delta_{\bar{p}} \kappa = 0$$

$$\delta h = 0 \qquad \delta_{\bar{p}} h = D_{\bar{p}}\kappa$$

$$\delta \kappa_{\bar{m}\bar{n}} = F_{\bar{m}\bar{n}} \qquad \delta_{\bar{p}} \kappa_{\bar{m}\bar{n}} = 0$$

$$\delta \bar{\Phi} = \bar{\kappa} \qquad \delta_{\bar{p}} \bar{\Phi} = 0$$

$$\delta \bar{\kappa} = 0 \qquad \delta_{\bar{p}} \kappa_{mn} = 2J_{\bar{p}[m} D_{n]} \bar{\Phi}$$

$$\delta \kappa_{mn} = B_{mn} \qquad \delta_{\bar{p}} B_{mn} = -2J_{\bar{p}[m} D_{n]} \bar{\kappa} + D_{\bar{p}} \kappa_{mn} - 2J_{\bar{p}[m}[\Psi_n, \bar{\Phi}]$$

$$\delta B_{mn} = 0 \qquad \delta_{\bar{p}} B_{\bar{m}\bar{n}} = 0$$

$$\delta B_{\bar{m}\bar{n}} = 2D_{[\bar{m}} \Psi_{\bar{n}]} + [\kappa_{\bar{m}\bar{n}}, \Phi] \qquad \delta_{\bar{p}} \Psi_{\bar{m}} = B_{\bar{p}\bar{m}}$$

$$\delta \Psi_{\bar{m}} = -D_{\bar{m}} \Phi \qquad \delta_{\bar{p}} \Phi = -\Psi_{\bar{p}}$$

$$\delta \Phi = 0 \qquad \delta_{\bar{p}} \kappa = D_{\bar{p}} \bar{\Phi}$$

Lagrangian :

The δ and $\delta_{\bar{p}}$ invariant action is made of two independent terms

$$\begin{aligned} L = \quad & \delta(\kappa_{\bar{m}\bar{n}} F_{mn} + \kappa(h + F_{m\bar{m}})) \\ + \quad & \delta(\kappa_{mn} B_{\bar{m}\bar{n}} + \bar{\Phi} D_m \Psi_{\bar{m}}) \end{aligned}$$

The $\delta_{\bar{p}}$ symmetry fixes the δ-antecedents of both terms, and, moreover, L is $\delta\delta_{\bar{p}}$-exact,

$$L = \int \delta\delta_{\bar{p}}(...)$$

Notice that one has the symmetry δ_{mn}, which pops up as a providential symmetry of both terms of $L = \int \delta\delta_{\bar{p}}(...)$, and satisfies

$$\{\delta_{mn}, \delta_{mn}\} = 0$$

$$\{\delta, \delta_{mn}\} = 0$$

$$\{\delta_{mn}, \delta_{\bar{p}}\} = J_{\bar{p}[m}\partial_{n]}$$

Once the existence of δ_{mn} has been noticed, the 4 generators $\delta, \delta_{\bar{p}}, \delta_{mn}$ can be untwisted as the 4 superpoincare generators $Q^{\alpha}, Q_{\dot{\alpha}}$.

Quantum field theory applications (not the subject here):

Only 3 generators are needed, for a complete analysis of all N=1,d=4 theories, including finiteness theorems, and the relationship between ABBJ anomalies and supersymmetry anomalies.

This generalizes, when on extends the supersymmetry, introducing R-symmetries. For instance, to prove the vanishing of the β function in N=4, or various N=2, only the invariance under 5 generators is needed to provide the necessary Ward identities, within the context of local quantum field theory.

A lot of this extends for superconformal symmetry in d=4; and also in d=8, using octonionic or $SU(4)$ type instantons.

N=1,d=4 supergravity revisited

$$I = \int \epsilon_{abcd}\ e^c e^d R^{cd}(\omega) \ + \ \bar{\lambda}\gamma_a e^a \gamma^5 D\lambda \qquad + \qquad B_2 dA + G_3^* G_3$$

A_1 and B_2 (Sokatchev and all; Stelle and West) are propagating auxiliary fields, with gauge invariances, giving an off-shell balanced multiplet(6,12,6)

$$(e^a(6 = 16 - 6 - 4), \lambda(12 = 16 - 4), A(3 = 4 - 1), B_2(3 = 6 - 4 + 1) = (6, 12, 6)$$

A gauges chirality. $D = \omega^{ab}\gamma_{ab} + A\gamma^5$ is the covariant derivative and

$$G_3 = dB_2 + \frac{1}{2}\bar{\lambda}\gamma^a\lambda e^a$$

The spin connection is constrained by

$$T^a = de^a + \omega^{ab} + \frac{1}{2}\bar{\lambda}\gamma^a\lambda = G^a_{cd}e^b e^c$$

so

$$\omega^{ab} = \omega^{ab}(e,\lambda,B_2) = \omega^{ab}(e,\lambda) + G^{ab}_{c}e^c$$

The gravitino $\lambda = \lambda_\mu dx^\mu$ is a 1-form Majorana spinor. It will be twisted accordingly.

Following a 1986 work of Baulieu and Bellon, call s the BRST operator of reparametrization (ξ), local susy(χ), Lorentz(Ω), chiral U(1) (c) , 2-form gauge symmetry (B^1_1). We can subtract from s the reparametrization, by defining $\hat{s}$

$$\hat{s} = s - \mathcal{L}_\xi$$

with

$$s\xi^\mu = \xi^\nu \partial_\nu \xi^\mu + \frac{1}{2}\bar{\chi}\gamma^\mu\chi$$

The off-shell closure relation $s^2 = 0$ is equivalent to

$$\hat{s}^2 = i_{\bar{\chi}\gamma\chi}$$

Reparametrization invariance is decoupled by the operation $exp - i_\xi$, when classical and ghost fields are unified into graded sums.

The action of the oeprator $\hat{s}$ is as follows

$$\begin{aligned}
\hat{s}e^a &= -\Omega^{ab}e^b - \lambda\gamma^a\chi \\
\hat{s}\lambda &= -D\chi - \Omega^{ab}\gamma^{ab}\lambda - c\gamma^5\lambda \\
\\
\hat{s}B_2 &= -dB_1^1 - \lambda\gamma^a\chi e^a \\
\hat{s}A &= -dc - \bar{\chi}\gamma^a\gamma^5 X^a \\
\\
\hat{s}\omega^{ab} &= -D\Omega^{ab} + \bar{\chi}\gamma^a\gamma^5 X^a
\end{aligned} \tag{1}$$

The spinor X^a turns out to be proportional to equations of motions,

$$X_a = \rho_{ab}e^b - (G_{abc}\gamma^{bc} + \epsilon_{abcd}\gamma^5)\lambda$$

The property $s^2 = 0$ is warranteed by the ghost transformation laws.

There is unification between classical fields and ghosts going on. It is as the one that occurs when analyzing anomalies by descent equations.

Here it is : $\omega \to \omega + \Omega$, $A \to A + c$, $\lambda \to \lambda + \chi$, $B_2 \to B_2 + B_1^1 + B_0^2$, $d \to d + \hat{s} + i_{\chi\gamma\chi}$. **The symmetry of supergravity is given by distorted horizontality conditions on the curvatures of unified fields, such as** $\hat{F} \equiv (d + \hat{s} + i_{\chi\gamma\chi})(A + c)$, **and so on.**

Off-shell closure is equivalent to the consistency of Bianchi identities.

The existence and the determination of the invariant action are almost obvious from the Bianchi identities.

One has in fact

$$\begin{aligned}
\hat{T}^a &\equiv \hat{d}e^a + (\omega+\Omega)^{ab}e^b + \frac{1}{2}(\bar{\lambda}+\bar{\chi})\gamma^a(\lambda+\chi) &&= G^a_{cd}e^b e^c \\
\hat{G}_3 &\equiv \hat{d}(B_2 + B_1^1 + B_0^2) + \frac{1}{2}(\bar{\lambda}+\bar{\chi})\gamma^a(\lambda+\chi)e^a &&= G_{abc}e^a e^b e^c \\
\hat{\rho} &\equiv (\hat{d}+\omega+\Omega+A+c)(\lambda+\chi) &&= \rho_{ab}e^a e^b \\
\hat{R}^{ab} &\equiv \hat{d}(\omega+\Omega) + (\omega+\Omega)^2 &&= R^{ab} + \bar{\chi}\gamma^{[a}X^{b]} + \frac{1}{2}\bar{\chi}\gamma^c\chi G^{ab}_c \\
\hat{F} &\equiv \hat{d}(A+c) &&= F + \bar{\chi}\gamma^a\gamma^5 X^a + \frac{1}{2}\bar{\kappa}\gamma^c\chi\epsilon_{abcd}G^{bcd}
\end{aligned} \qquad (2)$$

where $\hat{d} = d + \hat{s} + i_{\chi\gamma\chi}$**.**

The "horizontality distorting" spinor functional $X_a = \rho_{ab}e^b - (G_{abc}\gamma^{bc} + \epsilon_{abcd}\gamma^5)\lambda$ **is uniquely determined by Bianchi identities.**

Novel feature : Twisted expression of N=1, d=4 supergravity

The twisted gravitino is made of the 3 one-forms

$$\Psi_m, \kappa_{\bar{m}\bar{n}}, \kappa$$

The Rarita–Schwinger action is

$$\kappa_{\bar{m}\bar{n}} \wedge e_n \wedge D\Psi_m + \kappa \wedge e_{\bar{m}} \wedge D\Psi_m$$

The Einstein action will turn out to be very simple in twisted variables.

Does it mean that the link between topological gravity and supergravity is as simple as that between NSR superstrings and the topological sigma model?

The twisted supergravity fields are

$fields$	$gh.n.$	$m.d.$	$fields:$	$gh.n.$	$m.d.$
e_m	0	$/$	A	0	$/$
$e_{\overline{m}}$	0	$/$	B_2	0	$/$
ψ_m	1	$/$	$\omega_{\bar{m}\bar{n}}$	0	$/$
$\kappa_{\overline{mn}}$	-1	$/$	ω_{mn}	0	$/$
κ	-1	$/$	$\omega_{m\bar{n}}$	0	$/$

(3)

The needed twisted spinorial field-equation dependent functions X^a that we found for the distortion of horizontality equations of supergravity with M. Bellon in our old untwisted work are now (LB, M. Bellon V. Rey)

$$X_p, \quad X_{\bar{p}}, \quad X_{p,m}, \quad X_{\bar{p},m}, \quad X_{p,\bar{m}\bar{n}}, \quad X_{\bar{p},\bar{m}\bar{n}}, \tag{4}$$

For what follows, the important observations (see Landau, $\int R = \int \Gamma\Gamma$) is that the Einstein Lagrangian, modulo a boundary term, is the square of the self-dual (or antiselfdual) part of the connection

$$\int L_E = \int \omega^{ab^+} \wedge e^b \wedge e^c \wedge \omega^{ac^+} \tag{5}$$

This comes from

$$L_E = \epsilon_{abcd} e^a e^b R^{cd} = e^a e^b (R^{ab^+} - R^{ab^-})$$

$$e^a e^b R^{ab} = e^a e^b (R^{ab^+} + R^{ab^-}) = d(...)$$

We will decompose this in (anti)holomorphic components.

For any given 2-form F, one has

$$(F_+)_{mn} = F_{mn} \qquad (F_+)_{\bar{m}\bar{n}} = F_{\bar{m}\bar{n}} \qquad (F_+)_{\bar{m}n} = J_{\bar{m}n}(J^{\bar{p}q}F_{\bar{p}q})$$

One thus finds the yet unfamiliar expression of the Einstein action

$$\int L_E = \int \Big(\omega_{\bar{m}\bar{n}} \wedge e_n \wedge e_{\bar{p}} \wedge \omega_{pm} + \omega \wedge (e_n \wedge e_m \wedge \omega_{\bar{m}\bar{n}} + e_{\bar{n}} \wedge e_{\bar{m}} \wedge \omega_{mn} \Big)$$

where $\omega \equiv J^{\bar{p}q}\omega_{\bar{p}q}$**.**

The analogy with YM is striking ; It suggests us how to decompose the BRST symmetry operator s into scalar, vector and (irrelevant) antiself-dual parts.

	δ	$\delta_{\overline{q}}$	δ_{mn}
e_p	ψ_p	$J_{\bar{q}p}\kappa$	0
$e_{\bar{m}}$	0	$\kappa_{\bar{q}\bar{m}}$	$J_{\bar{m}[n}\psi_{m]}$
ψ_n	0	$\omega_{\bar{q}n} + J_{\bar{q}n}\omega$	0
κ	$\omega + A$	0	ω_{mn}
$\kappa_{\bar{m}\bar{n}}$	$\omega_{\bar{m}\bar{n}}$	0	$J_{\bar{m}[n}\omega_{m]\bar{n}} + J_{\bar{m}[n}J_{m]\bar{n}}\omega$
B_2	$\Psi_m e_{\bar{m}}$	$\kappa_{\bar{q}\bar{n}}e_n + J_{\bar{q}n}\kappa e_{\bar{n}}$	$\Psi_m e_n$
A	$X_{\bar{q},p}$	$X_{\bar{q}} + X_{m,\bar{m}\bar{q}}$	$X_{m,n}$
ω_{rs}	$X_{r,s}$	$J_{\bar{q}r}X_s$	0
$\omega_{\bar{r}\bar{s}}$	0	$X_{\bar{q},\bar{r}\bar{s}} + J_{\bar{q}q}X_{\bar{q},\bar{r}\bar{s}}$	$J_{\bar{r}[m}X_{\bar{s},n]}$
$\omega_{r\bar{s}}$	$X_{\bar{s},r} + J_{\bar{s},r}X_{\bar{q},p}$	$X_{\bar{q},r\bar{s}} + J_{\bar{q}r}X_{\bar{s}}$	0

(6)

The four twisted supersymmetry operators $\delta, \delta_{\bar{p}}$ and δ_{mn} are given by this table.

Asking $\delta, \delta_{\bar{p}}$ invariances, one finds that the supergravity action is thus localized around gravitational instantons, in a typically TQFT invariant way

$$I_E = \int \delta\Big(\kappa_{\bar{m}\bar{n}}\ e_n\ e_{\bar{q}}\ \omega_{pm} + \kappa(e_{\bar{n}}\ e_{\bar{m}}\ \omega_{nm} + e_n\ e_m\ \omega_{\bar{m}\bar{n}} + e_m\ e_{\bar{m}}\ \omega\ )\Big)$$

That is

$$I_E = \int\ \Big(\epsilon_{abcd}e^a e^b e^c e^d R^{cd}(\omega) + \bar{\lambda}\gamma_a e^a \gamma^5 D\lambda + B_2 dA + G_3^* G_3\Big)$$

where $T^a = G^a_{bc} e^b e^c$, that is $\omega^{ab} = \omega^{ab}(e, \lambda) + G^{ab}_c e^c$.

It is remarkable that everything has been fixed by the δ and $\delta_{\bar{p}}$ symmetries and the demand of a scaling dimension of the action identical to that of the Einstein Lagrangian. The underlying localization around gravitational instantons could help for computing certain topological quantities.

(I expect similar results in d=8, which, in turn, can be dimensionally reduced, $SU(4) \rightarrow SU(2)$.)

Overall conclusion

The $SU(N)$ decompositions of spinors enlighten quite well the construction of twisted supersymmetric theories and the questions of non-closure of supersymmetry. It gives efficient tools for proving supersymmetric QFT properties.

The trick is

$$Twisted\ (more\ geometrical)\ supersymmetry \quad \rightarrow \quad Poincare\ supersymmetry$$

Poincaré (as well as conformal) supersymmetry has a lot of troublesome redundancy. It may appear as a kind of physical effective symmetry, which sometimes emerges after untwisting a TQFT, whenever it is possible.